GOIDI AMERICAN JOURNAL

Journal
Goidi American Journal
of Innovation Development and Investment
GOIDI INTERNATIONAL GROUP OF INSTITUTION

# Numerical Integration by using interpolation formulas

**Oras B. Jamil**

Department of Mathematics, College of Education for pure science,

Al-Muthanna University, Samawah, Iraq

orasbj@mu.edu.iq

**Shahad Kazem Aweez**

**Abstract**

Numerical integration plays very important role in mathematics. In this research, overviews on the most common Numerical integration methods, namely, trapezoidal Simpson's1/٣rule, Simpson's 1/ 8rule and Weddle's rule. Different procedures compared and tried to evaluate the value of some definite integrals. A combined approach of different integral rules has been proposed for a definite integral to get more accurate value for all cases.

This paper describes classical quadrature method for the numerical solution by using polynomial with some interpolation formulas in numerical integration.

***Keywords***: Numerical integration; Classical quadrature formula; Trapezoidal rule; Simpson's 1/3 rule; Boole's rule; Weddle's rule.

## 1 Introduction

Throughout this paper, let k(t) be continuous on the interval $[t_1; t_2]$. Then evaluation of the definite integral maybe practically impossible or at least difficult, in this two cases we tried to solve it by numerical integration. N.I. is that the study of how the numerical value of an integral are often found. It is also called as quadrature which refers to finding a square whose area is the same as the area under the curve. The inverse process to differentiation in calculus is represented by

$$I = \int_{t_1}^{t_2} k(t)dt \qquad (1.1)$$

Which means that the integral of a function k(t) with respect to the independent variable t evaluated between the initial values $t = t_1$ and $t = t_2$ (Equation (1.1)).

GOIDI AMERICAN JOURNAL
Journal
Goidi American Journal
of Innovation Development and Investment
GOIDI INTERNATIONAL GROUP OF INSTITUTION

In other word, it is mean evaluate the area under the curve of the function k(t) between the two vertical lines $t = t_1$ to $t = t_2$ and the x-axis. We will show that graphically in fig1.

In the integration, the problem is how to evaluate the shaded area.

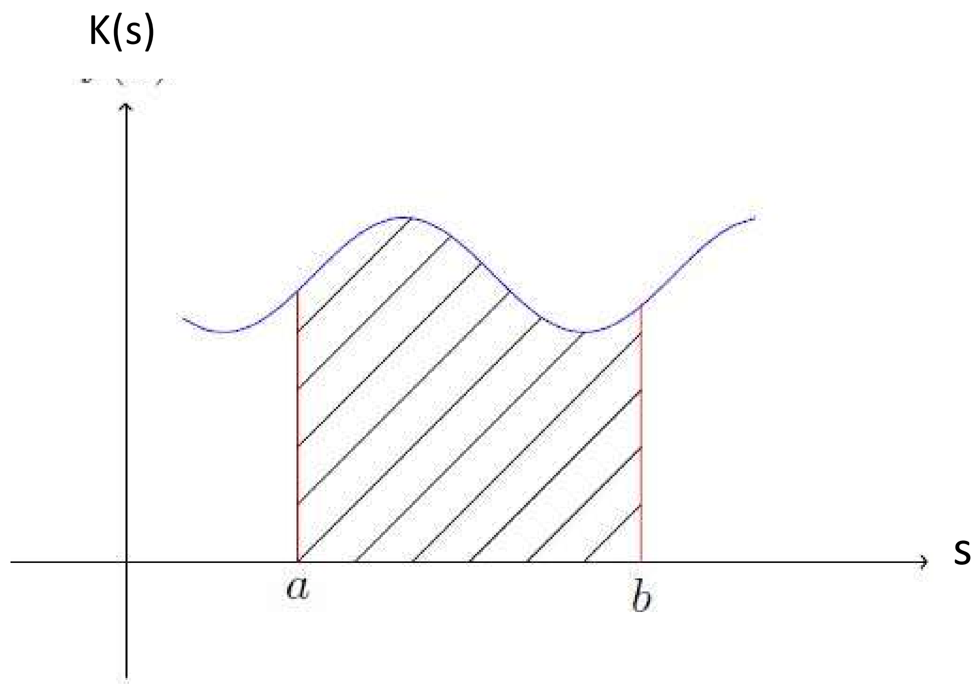


Figure 1: Graphical representation of integral of a function k(s)

The problem of integration is just reduced to the matter of finding shaded area. A better alternative approach might be to use a way that uses simple arithmetic operations to compute area. This approach is named as N.I. or numerical quadrature. N.I.M. uses an interpolating polynomial (I.P.) $P_n(t)$ in the place of k(t) which can be integrated analytically. We have seen several ways of numerical integration which replace the function k(t) by straight line or polynomial $P_n(t)$. In this paper, we interest by the polynomial $P_7(t)$ of order 7.

Thus

$$I = \int_{t_1}^{t_2} k(t)dt = \int_{t_1}^{t_2} P_7(t)dt \qquad (1.2)$$

GOIDI AMERICAN JOURNAL
Journal
Goidi American Journal
of Innovation Development and Investment
GOIDI INTERNATIONAL GROUP OF INSTITUTION

After this brief introduction, we include a section of preliminaries and notation re-garding Newten-Gregory forward interpolation formula. We also, review some rule which have been deriving from this formula like Trapezoidal, Simpson's 1/3 , Simpson's 3/8 , Boole and Weddle. In Section 3 we derive the 7B rule from this formula and we generalize it in Section 4. Finally, in Section 5 we discuss and comparing some problems.

## 2 Preliminaries

In this section, we use the general formula for solving N.I. it is also called general quadrature formula.

**General Quadrature Formula.**

Let $I = \int_a^b g\, dg$ , where $g$=k(s). Let k(s) be given for certain equidistant values of $s = s_0, s_0 + 2h,\ldots,s_0 + kh$

Suppose $g_0, g_1, \ldots \ldots g_k, k$ are the entries corresponding to the arguments

$s_0 = a, s_1 = a + h, s_2 = a + 2h, \ldots., s_k + a + kh = b$

respectively. Then we obtain, [3]

$$I = \int_a^b g\, dg = \int_{s_0}^{s_0+kh} g_s ds$$

We know,$u = \frac{s-s_0}{h}$

Or $s = s_0 ds = hdu$

Limits: When $s = s_0$, then $u = 0$

When $s = s_0 + kh$ , then $u = k$

$$I = \int_{x_0}^{x_0+kh} y \quad dx = \int_0^k y_{x_0+uh} hdu$$

$$= h \int_0^k \left[g_0 + u\Delta g_0 + \frac{u(u-1)}{2!}\Delta^2 g_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 g_0 + \cdots + \frac{u(u-1)(u-n+)}{n!}\Delta^k g_0\right] du$$

GOIDI AMERICAN JOURNAL

Journal
Goidi American Journal
of Innovation Development and Investment
GOIDI INTERNATIONAL GROUP OF INSTITUTION

$$= [kg_0 + \frac{k^2}{2}\Delta g_0 + \left(\frac{k^3}{3} - \frac{k^2}{2}\right)\frac{\Delta^2 g_2}{2!} + \left(\frac{k^2}{4} - k^3 + k^2\right)\frac{\Delta^3 g_0}{3!} + \left(\frac{k^5}{5} - \frac{3k^4}{2} + \frac{11k^3}{3} - 3k^2\right)\frac{\Delta^4 g_0}{4!} + \cdots ]$$

This is the required Newton-Cotes method, general quadrature formula. When k = 1, 2, 3......

then we obtain the Trapezoidal rule, Simpson's 1/3 rule, Simpson's 3/8 rule respectively. There are some graphical examples of Newton-Cotes where the integrating function can be polynomials for any order-for instance, (a) straight lines or (b) parabolas. The integral can be approximated in one step or in a series of steps to develop accuracy as [3]

Choosing interval size h = (b-a)/n, divide the interval [a, b] in to n intervals by means (n + 1) equally spaced point $t_0 = a$, $t_n = b$, $t_i = t_0 + ih$, i = 1, 2, …, n - 1

and let $g_i = k(t_i)$ for i = 0, 1, 2, …, n, the basic idea as we have seen in numerical integration is to replace the unknown tabulated function g = k(t) by an nth degree polynomial $P_n(t)$ say Newten-Gregory forward interpolation formula and carry on the integration. Thus

$$I = \int_{a=t_0}^{b=t_n} k(t)\,dt$$

$$\approx \int_{t_0}^{t_n} P_n(t)\,dt$$

$$= h\int_{c=0}^{n}\begin{bmatrix} g_0 + c\Delta g_0 + \frac{c(c-1)}{2}\Delta^2 g_0 + \frac{c(c-1)(c-2)}{3}\Delta^3 g_0 \\ + \ldots + \frac{c(c-1)(c-2)\ldots(c-n+1)}{n}\Delta^n g_0 \end{bmatrix} dc \ldots\ldots..(2.1)$$

Here the new variable is $c = \frac{t - t_0}{h}$, so $dc = \frac{dt}{h}$ where $h = \frac{b-a}{n}$

By changing n to different values various formulae is used to solve N.I. they are:

(i) If n = 1, then derivation of Trapezoidal rule formula.

(ii) If n = 2, then derivation of Simpson's 1/3 rd rule formula.

(iii) If n = 3, then derivation of Simpson's 3/8 th rule formula.

GOIDI AMERICAN JOURNAL
Journal
Goidi American Journal
of Innovation Development and Investment
GOIDI INTERNATIONAL GROUP OF INSTITUTION

(iv)If n = 4, then derivation of Boole's rule formula.

(v) If n = 6, then derivation of Weddle's rule formula.

In this paper, we mainly focus on to demonstrating of the rule when n = 7 in N.I.

**3 Main results**

We start by using a function g = k(s) on the interval [$s_0$, $s_7$], and put the equation (1.2) in the equation (2.1) by taking n = 7, this means we have eight points and seven intervals, so the equation (2.1) becomes:

$$\int_{s_0}^{s_7} g ds = h\int_0^7 [g_{0+} c\Delta g_0 + \frac{c(c-1)}{2!}\Delta^2 g_0 + \frac{c(c-1)(c-2)}{3!}\Delta^3 g_0 + \frac{c(c-1)(c-2)(c-3)}{4!}\Delta^4 g_0 + \frac{c(c-1)(c-2)(c-3)(c-4)}{5!}\Delta^5 g_0 + \frac{c(c-1)(c-2)(c-3)(c-4)(c-5)}{6!}\Delta^6 g_0 + \frac{c(c-1)(c-2)(c-3)(c-4)(c-5)(c-6)}{7!}\Delta^7 g_0]dc$$

$$= h\int_0^7 [g_0 + c\Delta g_0 + \left(\frac{c^2}{2} - \frac{c}{2}\right)\Delta^2 g_0 + \left(\frac{c^3}{6} - \frac{c^2}{2} + \frac{c}{3}\right)\Delta^3 g_0 + \left(\left(\frac{c^4}{24} - \frac{c^3}{4} + \frac{11c^2}{24} - \frac{c}{4}\right)\Delta^4 g_0 + \left(\frac{c^5}{120} - \frac{c^4}{12} + \frac{7c^3}{24} - \frac{5c^2}{12} + \frac{c}{5}\right)\Delta^5 g_0\right) + \left(\frac{c^6}{720} - \frac{c^5}{48} + \frac{17c^4}{144} - \frac{5c^3}{16} - \frac{137c^2}{360} - \frac{c}{6}\right)\Delta^6 g_0 + \left(\frac{c^7}{5040} - \frac{c^6}{240} + \frac{5c^5}{144} - \frac{7c^4}{48} + \frac{29c^3}{90} - \frac{7c^2}{20} + \frac{c}{7}\right)\Delta^7 g_0]dc$$

$$= h[g_0 c + \frac{c^2}{2}\Delta g_0 + \left(\frac{c^3}{6} - \frac{c^2}{4}\right)\Delta^2 g_0 + \left(\frac{c^4}{24} - \frac{c^3}{6} + \frac{c^2}{6}\right)\Delta^3 g_0 + \left(\frac{c^5}{120} - \frac{c^4}{16} + \frac{11c^3}{72} - \frac{c^2}{8}\right)\Delta^4 g_0 + \left(\frac{c^6}{720} - \frac{c^5}{60} + \frac{7c^4}{96} - \frac{5c^3}{36} + \frac{c^2}{10}\right)\Delta^5 g_0 + \left(\frac{c^7}{5040} - \frac{c^6}{288} + \frac{17c^5}{720} - \frac{5c^4}{64} + \frac{137c^3}{1080} - \frac{c^2}{12}\right)\Delta^6 g_0$$

GOIDI AMERICAN JOURNAL

Journal
Goidi American Journal
of Innovation Development and Investment
GOIDI INTERNATIONAL GROUP OF INSTITUTION

$$+\left(\frac{c^8}{40320}-\frac{c^7}{1680}+\frac{5c^6}{864}-\frac{7c^5}{240}+\frac{29c^4}{360}-\frac{7c^3}{60}+\frac{c^2}{14}\right)\Delta^7 g_0\Big]_0^7$$

$$= h\Big[7g_0+\frac{49}{2}\Delta g_0+\frac{539}{12}\Delta^2 g_0+\frac{1225}{24}\Delta^3 g_0+\frac{26117}{720}\Delta^4 g_0 + \frac{2499}{160}\Delta^5 g_0+\frac{30919}{8640}\Delta^6 g_0+\frac{5257}{17280}\Delta^7 g_0\Big]$$

$$=\frac{h}{17280}[120960g_0+423360\,(g_1-g_0)+776160(g_2-2g_1+g_0) + 882000(g_3-3g_2+3g_1-g_0)+626808(g_4-4g_3+6g_2-4g_1+g_0) + 269892(g_5-5g_4+10g_3-10g_2+5g_1-g_0) +161838(g_6-6g_5+15g_4-20g_3+15g_2-6g_1+g_0) +5257(g_7-7g_6+21g_5-35g_4+35g_3-21g_2+7g_1-g_0)\,]$$

$$=\frac{h}{17280}[5257g_0+25039g_1+9261g_2+20923g_3+20923g_4 + 9261g_5+25039g_6+\ 5257g_7]$$

$$=\frac{7h}{17280}[751g_0+3577g_1+1323g_2+2989g_3+\ 2989g_4 +1323g_5+3577g_6+\ 751g_7].$$

Put $B=\frac{1}{17280}$ in above equation. That is,

$$\int_{s_0}^{s_7} g\,ds = 7Bh[751g_0+3577g_1+1323g_2+2989g_3+2989g_4 +1323g_5+3577g_6+751g_7]$$

So, we name equation (3.2) as 7B rule formula.

## 4 Composite formula

Now, we generalize this result by using the composite integration methods. We divide the interval [a, b] into a number of sub-intervals and evaluate the integral in each sub-interval by the 7B rule formula.

We can derive composite formulae from average rule. If the range of integration is from a to a + nh = b, then this rule can be improved by dividing the interval

GOIDI AMERICAN JOURNAL
Journal
Goidi American Journal
of Innovation Development and Investment
GOIDI INTERNATIONAL GROUP OF INSTITUTION

[a, b] into sub-intervals of width 7h and apply 3.2 for each of the sub-interval. The sum of areas of all sub-intervals is the integral of the interval [a, b].Thus equation 3.2 becomes:

$$\int_{s_0}^{s_7} g\,ds = \int_{s0}^{s7} g\,ds + \int_{s7}^{s14} g\,ds + \cdots \ldots\ldots\ldots\ldots.. \int_{sn-7}^{sn} g\,ds$$

$$7\text{Bh}\begin{bmatrix} 751g_0 + 3577g_1 + 1323g_2 + 2989g_3 + 2989g_4 + 1323g_5 + \\ 3577g_6 + 751g_7 \end{bmatrix}$$

$$+7\text{Bh}[751g_7 + 3577g_8 + 1323g_9 + 2989g_{10} + 2989g_{11} + 1323g_{12} + 3577g_{13} + 751g_{14}]$$

$$7\text{Bh}[751g_{n-7} + 3577g_{n-6} + 1323g_{n-5} + 2989g_{n-4} + 2989g_{n-3} + 1323y_{n-2} + 3577g_{n-1} + 751g_n]$$

$$=7\text{Bh}[751(g_0 + g_n) + 1502(g_7 + g_{14} + \cdots \ldots\ldots. + g_{n-7}) + 3577(g_1 + g_6 + g_8 + g_{13} + g_{n-6} + g_{n-1}) + 1323(g_2 + g_5 + g_9 + g_{12} + g_{n-5} + g_{n-2}) + 2989(g_3 + g_4 + g_{10} + g_{11} + g_{n-4} + g_{n-3})] \qquad (4.1)$$

**5 Problems**

Problem 5.1. Using table 1we will find the approximate value of $\int_1^{11} \frac{ds}{s}$

Also, we find the exact solution and find the solution by using other method. Finally, we

| x | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | 1/2 | 1/3 | 1/4 | 1/5 | 1/6 | 1/7 | 1/8 | 1/9 | 1/10 | 1/11 |

Table 1: value of s and g=k(s) =1/s

Solution: We have a = 1, b = 11 and 10 intervals, then h = (b-a)/n= (11-1)/10=1.

Thus we use (4.1) to find the Solution:

GOIDI AMERICAN JOURNAL
Journal
Goidi American Journal
of Innovation Development and Investment
GOIDI INTERNATIONAL GROUP OF INSTITUTION

$$I = \int_{s_0=1}^{s_{10}=11} \frac{1}{s} ds$$

$$= 7Bh[751(g_0 + g_n) + 1502(g_7 + g_{14} + \cdots + g_{n-7}) + 3577(g_1 + g_6 + g_8 + g_{13} + \cdots + g_{n-6} + g_{n-1}) + 1323(g_2 + g_5 + g_9 + g_{12} + \cdots + g_{n-5} + g_{n-2}) + 2989(g_3 + g_4 + g_{10} + g_{11} + \cdots + g_{n-4} + g_{n-3})]$$

$$= \frac{7}{17280}[751(g_0 + g_{10}) + 1502(g_7) + 3577(g_1 + g_6 + g_8) + 1323(g_2 + g_5 + g_9) + 2989(g_3 + g_4 + g_{10}]$$

$$= \frac{7}{17280}\left[751\left(1 + \frac{1}{11}\right) + 1502\left(\frac{1}{8}\right) + 3577\left(\frac{1}{7} + \frac{1}{9} + \frac{1}{2}\right) + 1323(\frac{1}{3} + \frac{1}{6} + \frac{1}{10}) + 2989(\frac{1}{4} + \frac{1}{5} + \frac{1}{11})\right]$$

$$= 2.47696$$

Now, we find the exact solution

$$\int_1^{11} \frac{ds}{s} = ln(s)\,|_1^{11} = ln(11) - ln(1) = 2.397895.$$

then, the absolute error of the above solution is: $e = |2.47696 - 2.397895| = 0.079015$.

The solution by using Trapezoidal rule is:

$$\int_{s_0=1}^{s_{10}=11} \frac{ds}{s} = \frac{h}{2}[(g_0 + g_n) + 2(g_1 + g_2 + \cdots + g_{n-1})]$$

$$= \frac{1}{2}\left[1 + \frac{1}{11} + 2(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\right]$$

$$= 2.474423$$

The absolute error of Trapezoidal rule is: $e = |2.47696 - 2.397895| = 0.079015$. **Using Simpson's $\frac{1}{3}$**

$$\int_{s_0=1}^{s_{10}=1} \frac{ds}{s} = \frac{h}{3}[g_0 + 4(g_1 + g_3 + g_5 + \cdots + g_{n-1}) + 2(g_2 + g_4 + g_6 + \cdots + g_{n-2}) + g_n]$$

GOIDI AMERICAN JOURNAL
Journal
Goidi American Journal
of Innovation Development and Investment
GOIDI INTERNATIONAL GROUP OF INSTITUTION

$$= \frac{1}{3}\left[1 + 4\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10}\right) + 2\left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9}\right) + \frac{1}{11}\right]$$

$$= 2.410726$$

$e = |2.410726 - 2.397895|$

$e = 0.012831$

**Using Simpson's $\frac{3}{8}$**

$$= \frac{3h}{8}[g_0 + 2(g_3 + g_6 + g_9 + \cdots) + 3(g_1 + g_2 + g_4 + \cdots) + g_n]$$

$$= \frac{3}{8}\left[1 + 2\left(\frac{1}{4} + \frac{1}{7} + \frac{1}{10}\right) + 3\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9}\right) + \frac{1}{11}\right]$$

$= 2.394358 \approx 2.39436$

$e = |2.394538 - 2.397895|$

$e = 0.003357$

**Weddle's rule**

$$\int_1^{11} \frac{1}{s} ds = \frac{3h}{10}[g_0 + 5g_1 + g_2 + 6g_3 + g_4 + 5g_5 + 2g_6 + 5g_7 + g_8 + 6g_9 + g_{10}].$$

$$\frac{3}{10}\left[1 + \frac{5}{2} + \frac{1}{3} + \frac{6}{4} + \frac{1}{5} + \frac{5}{6} + \frac{2}{7} + \frac{5}{8} + \frac{1}{9} + \frac{6}{10} + \frac{1}{11}.\right]$$

e=[2.4231 − 2.397895] =0.025205

**Classical quadrature formula**

$$I = \int_{s_0=1}^{s_{10}=11} \frac{ds}{s}$$

Solution: given data $a = 1$, $b = 11$ and $k(s) = \frac{1}{s}$ Gauss Quadrature 2-point formula

$$I = \frac{b-a}{2}[w_1 k(s_1) + w_2 k(s_2)]$$

Where, $s_1 = \frac{b-a}{2} Z_1 + \frac{b+a}{2}$

GOIDI AMERICAN JOURNAL

Journal
Goidi American Journal
of Innovation Development and Investment
GOIDI INTERNATIONAL GROUP OF INSTITUTION

$$s_2 = \frac{b-a}{2} Z_2 + \frac{b+a}{2}$$

$$w_1 = w_2 = 1$$

$$Z_1 = \frac{-1}{\sqrt{3}} \quad , \quad Z_2 = \frac{1}{\sqrt{3}}$$

$$s_1 = \frac{11-1}{2}\left(\frac{-1}{\sqrt{3}}\right) + \frac{11+1}{2} = \frac{-5\sqrt{3}}{3} + 6$$

$$s_2 = \frac{11-1}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{11+1}{2} = \frac{5\sqrt{3}}{3} + 6$$

$$\mathrm{I} = \frac{b-a}{2}\left[w_1 k\left(\frac{-5\sqrt{3}}{3} + 6\right) + w_2 k\left(\frac{5\sqrt{3}}{3} + 6\right)\right]$$

$$\mathrm{I} = \frac{11-1}{2}\left[(1)\left(\frac{1}{\frac{-5\sqrt{3}}{3} + 6}\right) + (1)\left(\frac{1}{\frac{5\sqrt{3}}{3} + 6}\right)\right]$$

$$= 2.168674$$

$$\mathrm{e} = |2.16867 - 2.397895| = 0.229225$$

## 6 Conclusion

Comparing Our Proposed Numerical Integration Method (7B rule) to the various numerical integration formulas from above using the absolute errors,

the estimates of the area under the curve is accurate as the estimates of the S.1/3 R., S. 3/8 R. and Trapezoidal rule.

## References

[1] Md. Amanat Ullah, *Numerical Integration and a Proposed Rule*, American Journal of Engineering Research, 4(9), (2015), 120-123.

[2] J. Casaletto, M. Pickett and J. Rice, *A comparison of some numerical integration programs*, ACM SIGNUM Newsletter, ACM New York, NY, USA, 4(3), (1969), 30-40.

GOIDI AMERICAN JOURNAL

Journal
Goidi American Journal
of Innovation Development and Investment
GOIDI INTERNATIONAL GROUP OF INSTITUTION

[3] Mahesh Chalpuri and J. Sucharitha, *Numerical integration by using straight line interpolation formula*, Global Journal of Pure and Applied Mathematics, 13(6), (2017), 2123-2132.

[4] Michael Evans and Tim Swartz, *Methods for approximating integrals in statistics with special emphasis on Bayesian integration problems*, Statist. Sci., Inst. Math. Statist., Bethesda, MD. 10(3), (1995), 254-272.

[5] Eugene Isaacson and Herbert Bishop Keller, *Analysis of numerical methods*, Courier Corporation, (2012).

[6] Li Kaishun and Lumb Peter *Reliability analysis by numerical integration and curve fitting*, Structural safety, Elsevier, 3(1), (1985), 29-36.

[7] Obaid S Hanan. Aprospective study of future academic skills of basic education leaders as a resident supervisor in light of corona pandemic .(ISSUE:43),BOHOUTH MAGAZINE,Pp:21-32. (2021).

[9] Felix O. Mettle, Enoch N.B. Quaye, Louis Asiedu and Kwasi A. Darkwah, *A proposed method for numerical integration*, Journal of Advances in Mathematics and Computer Science, 17(1), (2016), 1-15.

[8] Rajesh Kumar Sinha and Rakesh Kumar, *Numerical method for evaluating the inte grable function on a finite interval*, International Journal of Engineering Science and Technology, 2(6), (2010).

*Using General Quadrature Formula*, Bulletin of Society for mathematical services & standards (B SO MA SS), 1(2), (2012), 1-5.